Listy Biometryczne – Biometrical Letters
Vol. 29 (1992), No. 2, 13-31

Testing for hypothesis in mixed linear models with two variance components

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Summary

In the paper a unified theory of testing for variance components in two variance component models is presented. Four equivalent approaches to the problem of reduction by invariance principle are discussed. The relations with the general approach of Kariya and Eaton are given. A test based on admissible estimators of variance components is given. Some invariant tests are presented as functions of maximal invariant statistics and relations between the tests are discussed. Necessary and sufficient conditions for existence of the uniformly most powerful invariant test are given. Examples of models that assure the existence of the uniformly most powerful invariant test are considered separately. For such examples the power functions of these tests are evaluated.

1. Introduction

We consider the following linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{U}\boldsymbol{\xi} + \mathbf{e} \quad , \tag{1.1}$$

where $\mathbf{y}$ is a normally distributed n-vector of observations, $\mathbf{X}$ is a known $n{\times}p$-matrix of the rank p, $\boldsymbol{\beta}$ is a p-vector of unknown fixed parameters, $\mathbf{U}$ is a known $n{\times}m$-matrix, $\boldsymbol{\xi}$ is an unobserved random m-vector with the expectation zero and with the variance covariance matrix $\sigma_1^2\mathbf{I}_m$, $\sigma_1^2 \geq 0$, while $\mathbf{e}$ is an n-vector of random errors which are assumed to be uncorrelated with the components of $\boldsymbol{\xi}$, and such that

$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta},\ \mathrm{Cov}(\mathbf{y}) = \sigma_1^2\mathbf{V} + \sigma^2\mathbf{I}_n,\ \mathbf{V} = \mathbf{U}\mathbf{U}',\ \sigma_1^2 \geq 0,\ \sigma^2 > 0\ . \tag{1.2}$$

Key words: mixed model, invariant tests for variances, two-way layouts.

We are interested in testing for the hypothesis

$$H:\ \sigma_1^2=0\ \text{ vs }\ K:\ \sigma_1^2>0 \tag{1.3}$$

or equivalently

$$H:\ \theta=0\ \text{ vs }\ K:\ \theta>0,\ \ \theta=\sigma_1^2/\sigma^2. \tag{1.4}$$

This problem has already been considered by several authors, usually for particular cases of the model (1.2) and under some additional assumptions on the structure of the variance-covariance matrix of **y**. The aim of this paper is to present a unified theory for testing the hypothesis (1.4). All tests we consider are invariant under a certain group of translations. We present four equivalent approaches to the problem of reduction of the model (1.2) by invariance principle that lead to construction of exact tests for testing (1.4). Necessary and sufficient conditions are given under which all the tests coincide with the uniformly most powerful invariant test (UMPIT). Examples of models corresponding to connected two-way layouts are considered separately. For such examples the power functions of these tests are evaluated.

2. Reduction of the model by invariance principle. The Neyman-Pearson's test

2.1 The approach of Olsen et al.

The problem of testing the hypothesis (1.4) is invariant under the group $\mathcal{G}_1$ of translations

$$g_\beta(\mathbf{y})=\mathbf{y}+\mathbf{X}\boldsymbol{\beta}\,.$$

Following Olsen, Seely and Birkes (1976) (see also LaMotte, 1976) a maximal invariant statistics with respect to $\mathcal{G}_1$ is **t=By**, where **B** is an $(n-p)\times n$-matrix defined as follows

$$\mathbf{BB}'=\mathbf{I}_{n-p},\quad \mathbf{B}'\mathbf{B}=\mathbf{M}\,.$$

Here $\mathbf{M}=\mathbf{I}-\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is the orthogonal projector on the kernel of $\mathbf{X}'$. The vector of expectations and the covariance matrix of **t** is given by

$$\mathrm{E}(\mathbf{t})=\mathbf{0},\ \ \mathrm{Var}(\mathbf{t})=\sigma_1^2\mathbf{W}+\sigma^2\mathbf{I}_{n-p},\quad \mathbf{W}=\mathbf{BUU}'\mathbf{B}'. \tag{2.1}$$

Denote by $\alpha_1>\alpha_2>\ldots\alpha_h\geq 0$ the ordered sequence of the different eigenvalues of **W** with the multiplicities $\nu_1,\ldots,\nu_h$. Let $\mathbf{W}=\sum_{i=1}^{h}\alpha_i\mathbf{E}_i$ be the spectral decomposition of $\mathbf{W}$ and $Z_i = \mathbf{t}'\mathbf{E}_i\mathbf{t}/\nu_i$, i=1,...,h. We shall assume through the paper that $h \geq 2$.

The following lemma established by Olsen et al. (1976) gives basic statistical properties of the statistic $\mathbf{Z} = (Z_1,...,Z_h)'$.

Lemma 2.1.

(i) $\nu_i Z_i \,/(\sigma_1^2\alpha_i + \sigma^2) \sim \chi^2_{\nu_i}$, $i = 1,2,...,h$,

(ii) $\mathbf{Z} = (Z_1,...,Z_h)'$ is a minimal sufficient statistics for the family of distributions of $\mathbf{t}$,

(iii) $\mathbf{Z}$ is a minimal complete statistics iff h=2,

(iv) $Z_1,...,Z_h$ are statistically independent,

(v) for an arbitrary $\mathbf{a} = (a_1, a_2,...,a_h)'$

$$\begin{cases} E(\mathbf{a}'\mathbf{Z}) = (\Sigma\, a_i\, \alpha_i)\sigma_1^2 + (\Sigma\, a_i)\sigma^2 \;, \\ \mathrm{Var}(\mathbf{a}'\mathbf{Z}) = 2\Sigma\, a_i^2(\sigma_1^2\alpha_i + \sigma^2)^2/\nu_i \;. \end{cases} \tag{2.2}$$

Let $\mathcal{G}_2$ be the group of transformations $g_a(\mathbf{Z}) = a^2\mathbf{Z}$, $a > 0$, defined on the set of sufficient statistics $\mathbf{Z}$. Then the problem of testing the hypothesis (1.4) is also invariant under the group $\mathcal{G}_2$. A maximal invariant statistics with respect to $\mathcal{G}_2$ is

$$\mathbf{Z}_o = (Z_1/Z_h,..., Z_{h-1}/Z_h)' \;.$$

The distribution function of $\mathbf{Z}_o$ is given in the next section (see also LaMotte et al., 1988).

2.2. Thompson's approach.

Thompson (1955) has considered the problem of testing the hypothesis (1.4) for the particular case of the model (1.2) corresponding to two-way classification. In application to the general model (1.2) the results of Thompson can be presented as follows. Let $\mathbf{C}$ be an $m\times m$-matrix given by

$$\mathbf{C} = \mathbf{U}'\mathbf{B}'\,\mathbf{B}\mathbf{U} = \mathbf{U}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{U} \;. \tag{2.3}$$

Note that $\mathbf{C}$ is the Schur's complement of $\mathbf{X}'\mathbf{X}$ in

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{U} \\ \mathbf{U}'\mathbf{X} & \mathbf{U}'\mathbf{U} \end{bmatrix}$$

(cf. Styan, 1983).

Denote by $\mathbf{K}$ the matrix whose the columns are the orthonormal eigenvectors of $\mathbf{C}$ corresponding to the positive eigenvalues, and let

$$\mathbf{Q} = \mathbf{U}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{y} = \mathbf{U}'\mathbf{M}\mathbf{y}\ ,$$

$$SS_W = \mathbf{y}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{y} = \mathbf{y}'\mathbf{M}\mathbf{y}\ ,$$

$$SS_o = SS_W - \mathbf{Q}'\mathbf{C}^{+}\mathbf{Q}\ ,$$

$$\mathbf{B}_o = \mathbf{X}'\mathbf{y}\ .$$

Theorem 2.1. (Thompson, 1955) The statistics $\mathbf{B}_o$, $\mathbf{G} = \mathbf{K}'\mathbf{Q}$ and SS_o are sufficient for the family of distributions of $\mathbf{y}$.

Thompson noted that the problem of testing the hypothesis (1.4) is invariant under the group $\mathcal{G}_o$ of transformations

$$B_{oj} \to aB_{oj} + a_j\ ,\ \mathbf{G} \to a\mathbf{G}\ ,\ SS_o \to a^2 SS_o\ ,$$

where $\mathbf{B}_o = (B_{o1}, B_{o2}, ..., B_{op})'$, a_j are arbitrary real numbers, while $a > 0$. If $SS_o > 0$, then a maximal invariant statistics with respect to $\mathcal{G}_o$ is

$$\mathbf{S} = (\sqrt{SS_o})^{-1}\mathbf{L}^{-1/2}\mathbf{G}\ , \tag{2.4}$$

where $\mathbf{L}$ is diagonal matrix with diagonal elements being the positive eigenvalues of $\mathbf{C}$.

To show an equivalence of the approaches of Thompson and Olsen et al. let us note that if $\mathbf{w}$ is the normalized eigenvector of $\mathbf{W} = \mathbf{B}\mathbf{U}\mathbf{U}'\mathbf{B}'$ corresponding to positive eigenvalue α, then

$$\mathbf{C}\mathbf{U}'\mathbf{B}'\mathbf{w} = \mathbf{U}'\mathbf{B}'\mathbf{B}\mathbf{U}\mathbf{U}'\mathbf{B}'\mathbf{w} = \alpha\mathbf{U}'\mathbf{B}'\mathbf{w} \quad \text{and} \quad \mathbf{w}'\mathbf{B}\mathbf{U}\mathbf{U}'\mathbf{B}'\mathbf{w} = \alpha\ .$$

It follows that $\mathbf{c} = (1/\sqrt{\alpha})\mathbf{U}'\mathbf{B}'\mathbf{w}$ is the normalized eigenvector of $\mathbf{C}$ corresponding to the eigenvalue α. Let $\mathbf{c}_{i1}, \mathbf{c}_{i2}, ..., \mathbf{c}_{i\nu_i}$ be normalized eigenvectors of $\mathbf{C}$ corresponding to α_i, i=1,2,...,d, where d=h−1 if $\alpha_h = 0$ and $d = h$ if $\alpha_h > 0$, while h is the number of different eigenvalues of $\mathbf{W}$. Let $\mathbf{C}_i = \sum_{l=1}^{\nu_i} \mathbf{c}_{il}\mathbf{c}'_{il}$. Then $\mathbf{C} = \sum_{i=1}^{d} \alpha_i \mathbf{C}_i$ is the spectral decomposition of $\mathbf{C}$.

Note that

$$\mathbf{Q}'\mathbf{C}_i\mathbf{Q} = (1/\alpha_i)\sum_{l=1}^{\nu_i} \mathbf{y}'\mathbf{B}'\mathbf{B}\mathbf{U}\mathbf{U}'\mathbf{B}'\mathbf{w}_i\mathbf{w}_i'\mathbf{B}\mathbf{U}\mathbf{U}'\mathbf{B}'\mathbf{B}\mathbf{y} = \alpha_i\mathbf{t}'\mathbf{E}_i\mathbf{t}\ ,\quad i = 1,...,d. \tag{2.5}$$

It follows that

$$(1/\alpha_i)\mathbf{Q}'\mathbf{C}_i\mathbf{Q} = \nu_i Z_i, \quad i = 1,\ldots,d \ ,$$

$$\mathbf{Q}'\mathbf{C}^{+}\mathbf{Q} = \sum_{i=1}^{d} \nu_i Z_i \ ,$$

$$SS_W = \mathbf{y}'\mathbf{M}\mathbf{y} = \sum_{i=1}^{h} \nu_i Z_i \ , \tag{2.6}$$

$$SS_o = SS_W - \mathbf{Q}'\mathbf{C}^{+}\mathbf{Q} = \begin{cases} \nu_h Z_h & \text{if } \ d = h-1 \quad (\alpha_h = 0) \\ 0 & \text{if } \ d = h \quad (\mathbf{W} > 0) \end{cases} .$$

Note that since $(n-p)\times(n-p)$-matrix $\mathbf{W}$ can be presented in the form $\mathbf{W} = \mathbf{W}_1\mathbf{W}'_1$, where $\mathbf{W}_1 = \mathbf{BU}$ is an $(n-p)\times m$-matrix and $\mathbf{C} = \mathbf{U}'\mathbf{B}'\mathbf{BU}$, then

$$\text{rank}(\mathbf{W}) = \text{rank}(\mathbf{C}) \ (= q, \text{ say})$$

$$n - p \geq q \ .$$

Besides $\nu_h = n-p-q$ is the multiplicity of zero eigenvalue of $\mathbf{W}$ and $\mathbf{W}$ is nonsingular ($SS_o = 0$) iff $n-p=q$. If we want to use the maximal invariant statistics $\mathbf{S}$ given by (2.4) we have to assume, that $n-p>q$. In section 6 examples for which $n-p=q$ are given. This case is also considered in section 4. To find the probability distribution function of the maximal invariant statistics $\mathbf{S}$ given by (2.4) note that

$$\mathbf{L}^{-1/2}\mathbf{G} \sim N_q\left\{0, \sigma^2(\mathbf{I} + \theta\mathbf{L})\right\}, \quad SS_o \sim \sigma^2\chi^2_{\nu_h},$$

where $\theta = \sigma_1^2/\sigma$, and that $\mathbf{L}^{-1/2}\mathbf{G}$ and SS_o are statistically independent. Hence the density of $\sqrt{\nu_h}\mathbf{S} = (SS_o/\nu_h)^{-1/2}\mathbf{L}^{-1/2}\mathbf{G}$ is the same as a q-component multivariate Student's t-statistic with ν_h degrees of freedom and with the location vector zero and the covariance matrix $\boldsymbol{\Sigma}(\theta) = \mathbf{I} + \theta\mathbf{L}$. After straightforward calculations it can be found that this density has the following form

$$f(\mathbf{s}) = (\pi\nu_h)^{-q/2}\,\frac{\Gamma((\nu_h+q)/2)}{\Gamma(\nu_h/2)}\,\frac{\sigma^{\nu_h}}{\sqrt{\det(\boldsymbol{\Sigma}(\theta))}}\,[1 + \mathbf{s}'\boldsymbol{\Sigma}^{-1}(\theta)\mathbf{s}]^{-(\nu_h+q)/2}.$$

Thus the probability distribution function of the maximal invariant statistics $\mathbf{S}$ is proportional to

$$[1 + \sum_{i=1}^{h-1}\sum_{l=1}^{\nu_i} s_{il}^2 \,/\, (1 + \alpha_i\,\theta)]^{-(\nu_h+q)/2} \ , \tag{2.7}$$

where

$$\mathbf{S} = (S_{11},..., S_{1\nu_1}, S_{21},..., S_{2\nu_2},..., S_{h-1,1},..., S_{h-1,\nu_{h-1}})',$$

while $S_{i1},..., S_{i\nu_i}$ correspond to eigenvalue α_i for $i=1,2,...,h-1$, $\sum_{i=1}^{h-1} \nu_i = q$ (cf. Thompson, 1955). Note that from (2.6)

$$\sum_{l=1}^{\nu_i} S_{il}^2 = (1/\alpha_i)\mathbf{Q}'\mathbf{C}_i\mathbf{Q}/SS_o = \nu_i Z_i / \nu_h Z_h , \quad i = 1,...,h-1 ,$$

and (2.7) becomes

$$[1 + \sum_{i=1}^{h-1} \nu_i Z_i / \nu_h Z_h (1 + \alpha_i \theta)]^{-(\nu_h + q)/2} =$$

$$= (\nu_h Z_h)^{(\nu_h + q)/2} [\sum_{i=1}^{h} \nu_i Z_i / (1 + \alpha_i \theta)]^{-(\nu_h + q)/2} . \tag{2.8}$$

Thus the ratio of the density of $\mathbf{S}$ at the null hypothesis $\mathrm{H}: \theta = 0$ and at the alternative $\mathrm{K}: \theta = \theta_*$ is

$$R = \left\{ \sum_{i=1}^{h} \nu_i Z_i / [\sum_{i=1}^{h} \nu_i Z_i / (1 + \alpha_i \theta_*)] \right\}^{-(\nu_h + q)/2} . \tag{2.9}$$

It follows that the test based on Neyman-Pearson's lemma for testing $\mathrm{H}: \theta = 0$ vs $\mathrm{K}: \theta = \theta_*$, $\theta_* > 0$, rejects $\mathbf{H}$ for sufficiently large

$$\mathrm{F}_{\mathrm{NP}}(\theta_*) = \sum_{i=1}^{h} \nu_i Z_i / [\sum_{i=1}^{h} \nu_i Z_i / (1 + \alpha_i \theta_*)] .$$

As it has been mentioned earlier, Thompson's approach needs the assumption that $\mathbf{W}$ is singular ($n>p+q$). Further we shall show that without any additional assumption on the model the Neyman-Pearson's test for $\mathrm{H}: \theta = 0$ vs $\mathrm{K}: \theta = \theta_*$, $\theta_* > 0$ is based on the test statistics $\mathrm{F}_{\mathrm{NP}}(\theta_*)$.

2.3. Mathew's approach

The reduction of observed vector $\mathbf{y}$ to maximal invariant statistics $\mathbf{S}$ proposed by Thompson (1955a) is done in two steps. In the first step the problem of testing the hypothesis (1.4) is reduced to sufficient statistics $\mathbf{B}_o$, $\mathbf{G}$ and SS_o (cf. Theorem 2.1). In the next step we find the maximal invariant statistic $\mathbf{S}$ with respect the group $\mathcal{G}_o$ of transformations on the set of sufficient statistics.

For the group $\mathcal{G}$ of transformations $\mathbf{y} \to a(\mathbf{y}+\mathbf{X}\boldsymbol{\beta})$ $a>0$, $\boldsymbol{\beta} \in R^p$, preserving the problem of testing the hypothesis (1.4) Mathew (1988) has considered a maximal invariant statistics of the form

$$\mathbf{T} = \mathbf{t} \,/\, \| \mathbf{t} \| = \mathbf{By}/\mathbf{y'My}\,, \tag{2.10}$$

where $\mathbf{t} = \mathbf{By}$, $\mathbf{B}$ is the matrix defined in the section 2.1, while $\| \mathbf{t} \| = \sqrt{\mathbf{t't}}$. Using the results of King (1980, Theorem 1) Mathew has given the following form for the ratio R of density functions of $\mathbf{T}$ under alternative $\mathrm{K}: \theta = \theta_*$ and under the null hypothesis $\mathrm{H}: \theta = 0$

$$\mathrm{R} = [\det(\boldsymbol{\Sigma}(\theta))]^{-1/2}\,[\det(\mathbf{X'X})]^{1/2}[\det(\mathbf{X'}\boldsymbol{\Sigma}(\theta)\mathbf{X})]^{-1/2} R_o\,,$$

where

$$R_o = \left[\frac{\mathbf{t'}(\mathbf{I}+\theta\mathbf{W})^{-1}\mathbf{t}}{\mathbf{t't}}\right]^{-(n-p)/2},$$

$$\boldsymbol{\Sigma}(\theta) = \mathbf{I}+\theta\mathbf{V} \quad \text{and} \quad \mathbf{W} = \mathbf{BVB'}\,.$$

It follows that the test based on the Neyman-Pearson's lemma for testing $\mathrm{H}: \theta = 0$ vs $\mathrm{K}: \theta = \theta_* > 0$ rejects H for sufficiently large $\mathbf{t't}/\mathbf{t'}(\mathbf{I}+\theta_*\mathbf{W})^{-1}\mathbf{t}$.

Using the spectral decomposition $\mathbf{W} = \sum_{i=1}^{h} \alpha_i \mathbf{E}_i$ it is easy to find that

$$(\mathbf{I}+\theta_*\mathbf{W})^{-1} = \sum_{i=1}^{h} (1+\theta_*\alpha_i)^{-1}\mathbf{E}_i$$

and

$$\mathbf{t't}/\mathbf{t'}(\mathbf{I}+\theta_*\mathbf{W})^{-1}\mathbf{t} = \sum_{i=1}^{h} \nu_i Z_i \,/\, [\sum_{i=1}^{h} \nu_i Z_i \,/\, (1+\alpha_i\theta_*)] = \mathrm{F_{NP}}(\theta_*)\,.$$

Note that this result has been obtained in section 2.2 under assumption that $\mathbf{W}$ is singular. Here this assumption is dropped.

2.4. General approach of Kariya and Eaton

The results presented in sections 2.1 to 2.3 on construction of the exact test for testing (1.4) can be obtained directly by using the general approach of Kariya and Eaton (1977).

Let $\mathcal{O}(n)$ be the group of $n{\times}n$ orthogonal matrices and let Q be the class of functions g such that

$$g: [0,\infty) \to [0,\infty),\ g \text{ is nonincreasing and } \int_{R^n} g(\mathbf{t't})d\mathbf{t} = 1\,.$$

For $\mathbf{x} \in R^n$ and for a fixed matrix $\mathbf{\Sigma}$ let

$$\mathcal{F}_o = \{ f:\ f(\mathbf{x}) = f(\mathbf{Gx}),\ \mathbf{G} \in O(n) \}\ ,$$

$$\mathcal{F}(\mathbf{\Sigma}) = \{ f:\ f(\mathbf{x}) = [\det(\mathbf{\Sigma})]^{-1/2}\, g(\mathbf{x\Sigma^{-1}x})\ ,\ \ g \in Q\ \}\ .$$

Let us note that $N(\mathbf{0}, \sigma^2\mathbf{I}) \in \mathcal{F}_o$, while $N(\mathbf{0}, \mathbf{\Sigma}) \in \mathcal{F}(\mathbf{\Sigma})$.

The following theorem has been proved by Kariya and Eaton (1977, Theorem 3.1).

Theorem 2.2. For a fixed $\mathbf{\Sigma}$ ($\mathbf{\Sigma} \neq \sigma^2\mathbf{I}$) the uniformly most powerful test for testing

$$\mathrm{H_o}: f \in \mathcal{F}_o \quad \text{vs} \quad \mathrm{K_o}: f \in \mathcal{F}(\mathbf{\Sigma}) \tag{2.11}$$

rejects $\mathrm{H_o}$ for large values of $\mathbf{x'x}/\mathbf{x'\Sigma^{-1}x}$. Under $\mathrm{H_o}$ the distribution of the test statistics $\mathbf{x'x}/\mathbf{x'\Sigma^{-1}x}$ is the same as that under assumption that $\mathbf{x} \sim N(\mathbf{0},\mathbf{I})$.

Remark 2.1. It is easy to note that theorem 2.2 is applicable if we replace $\mathbf{\Sigma}$ by $\lambda\mathbf{\Sigma}$, $\lambda > 0$.

In particular case $\mathbf{x} = \mathbf{t}$, where $\mathbf{t}$ has the normal distribution $N(\mathbf{0}, \mathbf{\Sigma}(\theta))$ with $\mathbf{\Sigma}(\theta) = \sigma^2(\mathbf{I} + \theta\mathbf{W})$, $\theta = \sigma_1^2/\sigma^2$, the hypothesis (2.11) reduces to

$$\mathrm{H}:\ \theta = 0 \quad \text{vs} \quad \mathrm{K}:\ \theta = \theta_*\ ,\ \ \theta_* > 0$$

and test which rejects H for sufficiently large

$$\mathrm{F_{NP}}(\theta_*) = \mathbf{t't}\ /\mathbf{t'}\,(\mathbf{I} + \theta_*\mathbf{W})^{-1}\mathbf{t}$$

coincides with the test based on Neyman-Pearson's lemma (see section 2.2 and 2.3).

Remark 2.2. More general theorem has been proved by Kariya and Sinha (1989) by using Anderson's representation theorem on the probability ratio of a maximal invariant (cf. Kariya and Sinha, 1989, pp. 65-67).

2.5. The uniformly most powerful invariant test

Mathew (1989) has noted that if the positive eigenvalues of **W** are all equal then the test based on Neyman-Pearson's lemma does ot depend on the alternative θ_*. It follows that in this case there exists the uniformly most powerful invariant test (UMPIT).

The following lemma gives a necessary and sufficient conditions that assure the existence of the UMPIT.

Lemma 2.2. There exists the UMPIT for testing

$$\mathrm{H}:\ \theta = 0 \quad \text{vs} \quad \mathrm{K}:\ \theta > 0$$

in the model (1.2) iff the number h of different eigenvalues of **W** is 2.

Proof. Let $\mathcal{C}(\theta_*)$ be the critical region of Neyman-Pearson's test for a fixed alternative K : $\theta = \theta_*$. It follows from the considerations in sections 2.2 and 2.3 that

$$\mathcal{C}(\theta_*) = \{\mathbf{Z} : \sum_{i=1}^{h} \nu_i Z_i > c(\theta_*) \sum_{i=1}^{h} \nu_i Z_i \,/\, (1+\alpha_i \theta_*)\},$$

where $c(\theta_*)$ is such that under H : $\theta = 0$

$$\Pr\{\mathbf{Z} \in \mathcal{C}(\theta_*)\} = \alpha, \quad \alpha > 0.$$

Note that

$$\mathcal{C}(\theta_*) = \{\mathbf{Z} : \sum_{i=1}^{h} a_i(\theta_*) Z_i > 0\}, \tag{2.12}$$

where

$$a_i(\theta_*) = [\nu_i(1+\alpha_i \theta_*) - c(\theta_*)\nu_i \,/\, (1+\alpha_i \theta_*)].$$

It is easy to see that $\mathcal{C}(\theta_*)$ given by (2.12) does not depend on θ_* iff

$$a_1(\theta_*) \,/\, a_h(\theta_*) = a_2(\theta_*) \,/\, a_h(\theta_*) = \ldots = a_{h-1}(\theta_*) \,/\, a_h(\theta_*).$$

The above condition is satisfied iff $c(\theta_*) = c(1+\alpha_i \theta_*)$ and h=2. The inverse implication is obvious.

Remark 2.3. The case h=2 and $\alpha_2 = 0$ is covered by Mathew's considerations. In section 6 examples of two-way layouts are given that lead to the model with h=2 and $\alpha_2 > 0$. For details concerning such models see Baksalary et al. (1990).

Remark 2.4. It has been discussed in Gnot and Kleffe (1983) that if h=2 then $\mathcal{W} = \mathrm{sp}\{\mathbf{I}_{n-p}, \mathbf{W}\}$ is a Jordan algebra, i.e. $\mathbf{A} \in \mathcal{W} \Rightarrow \mathbf{A}^2 \in \mathcal{W}$. In such a case according to the theory of Seely (1971) for each function $\mathbf{f}'\sigma$, there exists the uniformly best invariant quadratic and unbiased estimator.

3.Wald's test

The considerations of Seely and El-Bassiouni (1983) lead to exact Wald's test for testing H given by (1.4).

This test is based on $F_W = \mathbf{y}'\mathbf{\Pi}_1\mathbf{y}/\mathbf{y}'\mathbf{\Pi}_2\mathbf{y}$, where $\mathbf{\Pi}_1$ and $\mathbf{\Pi}_2$ are the orthogonal projectors on $\mathcal{R}(\mathbf{X} \vdots \mathbf{U}) \cap \mathcal{N}(\mathbf{X}')$ and $\mathcal{N}(\mathbf{X}') \cap \mathcal{N}(\mathbf{U}')$, respectively.

Lemma 3.1.

(i) Under H, $\mathbf{y}'\mathbf{\Pi}_1\mathbf{y}/\sigma^2 \sim \chi^2_k$, where $k = \text{rank}(\mathbf{X} \vdots \mathbf{U}) - \text{rank}(\mathbf{X})$,

(ii) $\mathbf{y}'\mathbf{\Pi}_2\mathbf{Y}/\sigma^2 \sim \chi^2_f$, where $f = n - \text{rank}(\mathbf{X} \vdots \mathbf{U})$, while n is the total number of observations,

(iii) $\mathbf{y}'\mathbf{\Pi}_1\mathbf{y}$ and $\mathbf{y}'\mathbf{\Pi}_2\mathbf{y}$ are statistically independent.

From Lemma 3.1 we find that under H the ratio $f\,\mathbf{y}'\mathbf{\Pi}_1\mathbf{y}/k\,\mathbf{y}'\mathbf{\Pi}_2\mathbf{y}$ has a central F-distribution with k and f degrees of freedom.

To get some geometrical interpretation of the above result let us define, for an arbitrary matrices **A** and **B** with the same number of rows, the following matrices

$\mathbf{P_A} = \mathbf{A}(\mathbf{A}'\mathbf{A})^-\mathbf{A}'$, (the orthogonal projector on $\mathcal{R}(\mathbf{A})$),

$\mathbf{M_A} = \mathbf{I} - \mathbf{P_A}$, (the orthogonal projector on $\mathcal{N}(\mathbf{A}')$),

$\mathbf{C_{A,B}} = \mathbf{A}'\mathbf{M_B}\,\mathbf{A}$, (Schur's complement of $\mathbf{A}'\mathbf{A}$ in

$$\begin{bmatrix} \mathbf{A}'\mathbf{A} & \mathbf{A}'\mathbf{B} \\ \mathbf{B}'\mathbf{A} & \mathbf{B}'\mathbf{B} \end{bmatrix}).$$

Lemma 3.2.

(i) $\mathbf{\Pi}_1 = \mathbf{M_X}\mathbf{U}\mathbf{C}^-_{\mathbf{U},\mathbf{X}}\mathbf{U}'\,\mathbf{M_X}$ is the orthogonal projector on $\mathcal{R}(\mathbf{X}|\mathbf{U}) \cap \mathcal{N}(\mathbf{X}') = \mathcal{R}(\mathbf{X}|\mathbf{U}) \cap \mathcal{R}^\perp(\mathbf{X})$,

(ii) $\mathbf{\Pi}_2 = \mathbf{M_X}(\mathbf{I} - \mathbf{U}\mathbf{C}^-_{\mathbf{U},\mathbf{X}}\mathbf{U}')\mathbf{M_X} = \mathbf{M_U}(\mathbf{I} - \mathbf{X}\mathbf{C}^-_{\mathbf{X},\mathbf{U}}\mathbf{X}')\mathbf{M_U}$ is the orthogonal projector on $\mathcal{N}(\mathbf{X}') \cap \mathcal{N}(\mathbf{U}') = \mathcal{R}^\perp(\mathbf{X} \vdots \mathbf{U}) = \mathcal{R}^\perp(\mathbf{U}) \cap \mathcal{R}^\perp(\mathbf{X})$,

(iii) $\mathbf{\Pi}_1 + \mathbf{\Pi}_2 = \mathbf{M_X}$

From lemma 3.2. and from (2.6) it follows, that $\mathbf{Q}'\mathbf{C}^+\mathbf{Q} = \mathbf{y}'\mathbf{\Pi}_1\mathbf{y}$ does not depend on the choice of the generalized inverse of $\mathbf{C}^-$. Besides $SS_\mathrm{W} = \mathbf{y}'(\mathbf{\Pi}_1 + \mathbf{\Pi}_2)\mathbf{y}$ and $SS_o = \mathbf{y}'\mathbf{\Pi}_2\mathbf{y}$. Thus Wald's test rejects H for sufficiently large

$$F_\mathrm{W} = \sum_{i=1}^{h-1} \nu_i Z_i \,/\nu_h Z_h\,.$$

It is easy to see that in the case h=2 this test coincides with the UMPIT given in section 2.5. Generaly taking $\theta_* \to \infty$ in the $F_\mathrm{NP}(\theta_*)$ we obtain

$$\lim_{\theta_* \to \infty} F_\mathrm{NP}(\theta_*) = \lim_{\theta_* \to \infty} \sum_{i=1}^{h} \nu_i Z_i \,/[\sum_{i=1}^{h} \nu_i Z_i \,/(1+\alpha_i\theta_*)] = \sum_{i=1}^{h} \nu_i Z_i \,/\nu_h Z_h\,,$$

which is a monotonic function of the ratio of $\sum_{i=1}^{h-1} \nu_i Z_i \,/\nu_h Z_h = F_\mathrm{W}$. Thus Wald's test is equivalent to the test based on $F_\mathrm{NP}(\theta_*)$ as θ_* tends to infinity.

Note that

$$\begin{bmatrix} X' \\ U' \end{bmatrix} [X\ U] = \begin{bmatrix} X'X & X'U \\ U'X & U'U \end{bmatrix} =$$

$$= \begin{bmatrix} I & 0 \\ U'X(X'X)^{+} & I \end{bmatrix} \begin{bmatrix} X'X & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} I & (X'X)^{+}X'U' \\ 0 & I \end{bmatrix}$$

and rank(**X′X**)=p. It follows that rank(**X** ⋮ **U**)= $p + q$, where q = rank(**C**) = rank(**X** ⋮ **U**)–rank(**X**), and the multiplicity of zero eigenvalues of **W** is $\nu_h = n-$ rank(**X** ⋮ **U**). Thus similarly as in Thompson's approach Wald's test by construction needs the assumption that **W** is singular ($n >$ rank(**X** ⋮ **U**)).

4. Test based on admissible estimators of variance components.

It follows from the consideration in section 3 and from (2.6) that under assumption $n >$ rank(**X** ⋮ **U**) ($\alpha_h = 0$) we have

$$\mathbf{y'\Pi_1 y} = \sum_{i=1}^{h-1} \nu_i Z_i \quad \text{and} \quad \mathbf{y'\Pi_2 y} = \nu_h Z_h \,. \tag{4.1}$$

Following (2.2) $E(\mathbf{y'\Pi_1 y}) = \mathrm{tr}(\mathbf{W})\sigma_1^2 + k\sigma^2$, k=rank(**W**), while $E(\mathbf{y'\Pi_2 y}) = \nu_h\sigma^2$, so that $\mathbf{y'\Pi_1 y}/k$ and $\mathbf{y'\Pi_2 y}/\nu_h$ are invariant and unbiased estimators of $[\mathrm{tr}(\mathbf{W})/k]\sigma_1^2 + \sigma^2$ and σ^2 respectively. It has been mentioned by Gnot and Michalski (1992) that $\mathbf{y'\Pi_1 y}/k$ and $\mathbf{y'\Pi_2 y}/\nu_h$ are admissible invariant quadratic and unbiased estimators of $[\mathrm{tr}(\mathbf{W})/k]\,\sigma_1^2 + \sigma^2$ and σ^2, respectively, iff $h = 2$.

In Gnot et al. (1985) a full characterization of nonnegative admissible invariant quadratic and unbiased estimators of $\rho\sigma_1^2 + \sigma^2$ is given for ρ such that

$$\rho_{\min} \le \rho \le \rho_{\max}\,,$$

where

$$\rho_{\min} = \begin{cases} \dfrac{\alpha_1 \mathrm{tr}(\mathbf{W}^{+}) - \mathrm{rank}(\mathbf{W})}{\alpha_1 \mathrm{tr}(\mathbf{W}^{+}\mathbf{W}^{+}) - \mathrm{tr}(\mathbf{W}^{+})} & \text{for} \quad \alpha_h > 0\,, \\ 0 & \text{for} \quad \alpha_h = 0\,. \end{cases}$$

$$\rho_{\max} = \frac{\mathrm{tr}(\mathbf{W}^2) - \alpha_h \mathrm{tr}(\mathbf{W})}{\mathrm{tr}(\mathbf{W}) - \alpha_h \mathrm{rank}(\mathbf{W})}\,.$$

We can obtain a modification of Wald's test taking as a test statistics the ratio of nonnegative admissible invariant quadratic and unbiased estimators. Intuitively, it seems to be reasonable to put as the numerator of the ratio the admissible and nonnegative estimator of $\rho_{max}\sigma_1^2 + \sigma^2$ and as the denominator, the admissible and nonnegative estimator of $\rho_{min}\sigma_1^2 + \sigma^2$. In consequence the test rejects the hypothesis H : $\theta = 0$ for sufficiently large

$$F_o = \begin{cases} \sum_{i=1}^{h}(\alpha_i - \alpha_h)\nu_i Z_i \,/ \sum_{i=1}^{h}(\alpha_1 - \alpha_i)\alpha_i^{-2}\nu_i Z_i & \text{for} \quad \alpha_h > 0\,, \\ \sum_{i=1}^{h-1}\alpha_i\,\nu_i Z_i \,/ \nu_h Z_h & \text{for} \quad \alpha_h = 0\,. \end{cases} \tag{4.2}$$

In the case when $\alpha_h = 0$ the above test has been proposed by Gnot and Michalski (1991).

Another test based on variance component estimators has been proposed by Michalski and Zmyślony (1992). At the construction of the test the authors used the known fact in theory of variance component estimation, that for each unbiased estimator $\mathbf{y'Ay}$ of σ_1^2 the matrix $\mathbf{A}$ can be decomposed as $\mathbf{A}_+ - \mathbf{A}_-$, where $\mathbf{A}_+$ and $\mathbf{A}_-$ are nonzero nonnegative definite matrices. In the paper the authors discuss the problem of an optimal choice of the estimator $\mathbf{y'Ay}$. One of them is MINQUE that leads to the test rejecting H : $\theta = 0$ for sufficiently large

$$F_1 = \mathbf{y'A}_+\mathbf{y} / \mathbf{y'A}_-\mathbf{y} = \sum_{\alpha_i^* > 0} \nu_i\,\alpha_i^* Z_i \;/ \sum_{\alpha_i^* < 0} -\nu_i\,\alpha_i^* Z_i\ , \tag{4.3}$$

where $\alpha_i^* = \alpha_i - \mathrm{tr}(\mathbf{W}) / \mathrm{rank}(\mathbf{W})$.

5. Locally best tests

If the UMPI test does not exist for testing hypothesis (1.4), we can find tests which are the best for alternatives close (in some sense) to the null hypothesis and hope that such tests will also be good for far alternatives. We consider the case, where the density function of $\mathbf{x}$ with respect to Lebesgue's measure μ on R^n is $f(\mathbf{x},\theta)$, where a single parameter θ is assumed to take values from an interval on the real line. The null hypothesis H says that $\theta = 0$. If ω is a critical region such that

$$\int_\omega f(\mathbf{x},\theta \,|\, \theta = 0)\mathrm{d}\mu = \alpha\,, \tag{5.1}$$

then the power function of the test is

$$p(\theta) = \int_\omega f(\mathbf{x},\theta)\mathrm{d}\mu\,.$$

Let $p(\theta)$ admit Taylor expansion around $\theta = 0$. Since $p(0) = \alpha$, we have

$$p(\theta) = \alpha + \theta p'(0) + o(\theta) \,.$$

If K is the class of one-sided alternatives $\theta > 0$, we need to maximize $p'(\theta)$ at $\theta = 0$ to obtain a locally best (most powerful) one-sided test. We will assume differentiation under the integral sign so that the quantity to be maximized is

$$p'(0) = \int_\omega f'(\mathbf{x},\theta|\,\theta = 0)\mathrm{d}\mu \,.$$

Lemma 5.1. [Rao (1973), p.454]

Let ω be any region such that

$$p(0) = \int_\omega f(\mathbf{x},\theta|\,\theta = 0)\mathrm{d}\mu = \alpha$$

and ω_o be the region

$$\{x: \;\; f'(\mathbf{x},\theta|\,\theta = 0) \;\geq\; k f(\mathbf{x},\theta|\,\theta = 0)\} \,,$$

where k is such that the condition (5.1) is satisfied for ω_o. Then

$$\int_{\omega_o} f'(\mathbf{x},\theta|\,\theta = 0)\mathrm{d}\mu \;\geq\; \int_\omega f'(\mathbf{x},\theta|\,\theta = 0)\mathrm{d}\mu \,.$$

The result is obtained by an application of Neyman-Pearson's lemma [cf. Rao (1973), p.446].

Now, applying lemma 5.1 to density function $f(\mathbf{s},\theta)$ of the maximal invariant statistics S from section 2 by straightforward calculations it can be checked that the locally best invariant test (LBIT) is based on the following statistics

$$F_{\mathrm{LB}} = \sum_{i=1}^{h-1} \alpha_i \, \nu_i \, Z_i \,/ \sum_{i=1}^{h} \nu_i Z_i = \mathbf{Q'Q} \,/ SS_{\mathrm{W}} \,.$$

Further, we prove that LBIT can be obtained in an equivalent way letting $\theta_* \to 0$ in the $F_{\mathrm{NB}}(\theta_*)$. Let us express the rejection region of the Neyman-Pearson's test

$$F_{\mathrm{NB}}(\theta_*) = \sum_{i=1}^{h} \nu_i Z_i \,/\, [\sum_{i=1}^{h} \nu_i Z_i \,/\, (1 + \alpha_i \theta_*)] \;>\; c_\alpha$$

as

$$[\theta_*^{-1} \sum_{i=1}^{h} \nu_i Z_i \,(1 - 1/(1 + \alpha_i \theta_*)]/[\sum_{i=1}^{h} \nu_i Z_i/(1 + \alpha_i \theta_*)] \;>\; c'_\alpha = (c_\alpha - 1)\,/\,\theta_* \,.$$

Hence

$$\lim_{\theta_* \to 0} F_{NB}(\theta_*) = \lim_{\theta_* \to 0} [\sum_{i=1}^{h} \nu_i Z_i \alpha_i \ / \ (1+\alpha_i \theta_*)] \ /[\sum_{i=1}^{h} \nu_i Z_i \ /(1+\alpha_i \theta_*)] =$$

$$= \sum_{i=1}^{h-1} \alpha_i \nu_i Z_i \ / \sum_{i=1}^{h} \nu_i Z_i = F_{LB} \ .$$

The above result has been obtained by Westfall (1989).

6. Examples.

Numerical comparison of the power functions of tests can be found in several papers. In unbalanced, random one-way ANOVA models LaMotte at al. (1988) compared the tests based on Neyman-Pearson's lemma with so called LM tests. These models have also been used by El-Bassiouni and Seely (1988) to study the efficiency of Wald's test. Gnot and Michalski (1992) and Michalski and Zmyślony (1992) have compared the power functions of all described here tests in the random models corresponding to block designs. In all the considered models the essential assumption that **W** is singular has been made. General conclusion is that at a fixed number of observations the properties of the test strongly depend on the multiplicity ν_h of zero eigenvalues of **W**.

In this section we give a numerical comparison of the power functions of the tests for models corresponding to two-way layouts with **W** nonsingular, i.e. when $n=\text{rank}(\mathbf{X}|\,\mathbf{U})$.

Let us consider an experiment in which n experimental units are arranged in a rows and b columns according to the $a\times b$ incidence matrix **N** with entries $n_{ij} \geq 0$.

It is assumed that a linear model corresponding to described above two-way layouts is the mixed model with random row effects, with fixed column effects and without interaction. It can be presented as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{U}\boldsymbol{\xi} + \mathbf{e} \ ,$$

where **y** is an $n\times1$ vector of observations, $\boldsymbol{\beta}$ is a $b\times1$ vector of unknown column effects, $\boldsymbol{\xi}$ is an $a\times1$ vector of random row effects with the expectation **0** and covariance matrix $\sigma_1^2\mathbf{I}_a$, while **e** is an $n\times1$ vector of errors with the expectation **0**, covariance matrix $\sigma^2\mathbf{I}_n$ and uncorrelated with $\boldsymbol{\xi}$. Besides **X** and **U** are $n\times b$ and $n\times a$ known (0,1)-matrices of full ranks b and a. It is clear that $\mathbf{U}'\mathbf{X} = \mathbf{N}$, $\mathbf{U}'\mathbf{U} = \mathbf{D}_r$ and $\mathbf{X}'\mathbf{X} = \mathbf{D}_c$, where $\mathbf{D}_r$ and $\mathbf{D}_c$ are diagonal matrices with the diagonal elements being the components of $\mathbf{r} = \mathbf{N}\mathbf{1}_b$ and $\mathbf{k} = \mathbf{N}'\mathbf{1}_a$, respectively, where $\mathbf{1}_z$ denotes the $z\times1$ vector of ones. Under these assumptions the expectation and the covariance matrix of **y** are

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}\,,\quad \mathrm{Cov}(\mathbf{y}) = \sigma_1^2\mathbf{U}\mathbf{U}' + \sigma^2\mathbf{I}_n\,. \tag{6.1}$$

In this special case of the general model (1.2) the orthogonal projector **M** on the kernel of **X**′, and the $a\times a$ matrix **C** given by (2.3) have the form

$$\mathbf{M} = (\mathbf{I}_n - \mathbf{X}\,\mathbf{D}_c^{-1}\mathbf{X}')\,,$$

$$\mathbf{C} = \mathbf{D}_r - \mathbf{N}\mathbf{D}_c^{-1}\mathbf{N}'\,.$$

Let rank(**C**) = q. A two-way layout with a rows and b columns is said to be connected if $q=a-1$. According to the considerations in section 2 for connected two-way layouts the following inequality holds

$$n \geq a + b - 1,$$

and if $n>a+b-1$, then $\nu_h=n-a-b+1$ is the multiplicity of the zero eigenvalues of **W**. Below two examples of layouts for which $n=a+b-1$ are given.

Example 1

$$\mathbf{N} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \\ 1&1&1&1&1&1&1&1&1 \end{bmatrix}$$

Example 2

$$\mathbf{N} = \begin{bmatrix} 1&0&0&0 \\ 1&1&0&0 \\ 0&1&0&0 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&1&0 \\ 0&0&1&1 \\ 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \end{bmatrix}$$

A detailed characterization of such layouts have been given by Kageyama (1985) (see also Baksalary et al., 1990, section 3).

In the first example the number h of different eigenvalues of $\mathbf{W}$ is 2, ($\alpha_1 = 5$, $\nu_1 = 1$; $\alpha_2 = 0.5$, $\nu_2 = 8$) and in consequence there exist the UMPIT for testing

$$H : \sigma_1^2 = 0 \quad \text{vs} \quad K : \sigma_1^2 > 0$$

(cf. lemma 2.2). In Table 1 the power function $p(\theta)$ of the UMPI test is presented as a function of $\theta = \sigma_1^2/\sigma^2$. .

Table 1

The power function of the UMPI test for example 1

θ	$p(\theta)$
.001	.0504
.010	.0540
.100	.0898
.500	.2055
1.000	.2822
5.000	.4221
10.000	.4518
50.000	.4793
100.000	.4830
500.000	.4860
∞	.4867

In the second example the UMPIT does not exist. The matrix $\mathbf{W}$ has 7 different eigenvalues

($\alpha_1 = 1.89$ $\nu_1 = 1$; $\alpha_2 = 1.67$ $\nu_2 = 1$; $\alpha_3 = 1.43$ $\nu_3 = 1$; $\alpha_4 = 1.00$ $\nu_4 = 4$; $\alpha_5 = 0.57$ $\nu_5 = 1$; $\alpha_6 = 0.33$ $\nu_6 = 1$; $\alpha_7 = 0.11$ $\nu_7 = 1$)

In Table 2 the power functions $p(\theta)$ of Neyman-Pearson's (NP) test based on F_{NP} statistic, locally best (LB) test based on F_{LB} ratio, the test (GM) based on the ratio F_o of admissible estimators and the test (ZM) based on F_1 are presented as functions of $\theta = \sigma_1^2/\sigma^2$. In the last column of the table the attainable upper bounds (AUB) of the power functions are presented. The values of AUB have been defined by LaMotte at al. (1988) as values of the power function of the NP test at $\theta = \theta_*$. The losses of power functions of the tests in comparing with the AUB are presented in the second column as natural numbers. This loss is calculated according to the formula

$$l(\theta) = 100\% \, [\mathrm{AUB}(\theta) - p(\theta)] \, / p(\theta)$$

Table 2
The power functions of the tests for example 2

θ	NP		LB		GM		ZM		AUB
.010	.0511	0	.0511	0	.0508	1	.0507	1	.0511
.050	.0556	0	.0556	0	.0538	3	.0535	4	.0556
.100	.0610	0	.0610	0	.0574	6	.0569	7	.0610
.500	.0987	0	.0981	1	.0836	15	.0789	20	.0988
1.000	.1344	1	.1316	2	.1100	18	.0980	27	.1344
5.000	.2290	10	.2251	11	.2138	16	.1507	41	.2540
10.000	.2582	14	.2530	16	.2619	13	.1666	45	.3013
50.000	.2890	20	.2822	22	.3308	9	.1834	50	.3638
100.000	.2934	21	.2864	24	.3433	8	.1859	50	.3746
500.000	.2974	22	.2899	25	.3541	8	.1879	51	.3840

The powers of the tests have been computed using a modified procedure of Imhof (1961). This procedure allows to compute the probability distribution function of an arbitrary linear combination of $\mathbf{t}'\mathbf{E}_i\mathbf{t}$, $i=1,2,\ldots,h$.

The inspection of Tables 1 and 2 shows a low power functions of all the tests even for far alternatives. It is caused by the equation $n=a+b-1$ and in consequence by the positive definitness of **W**. Below an example of the so called variance balanced design is given (example 3).

Example 3

$$\mathbf{N} = \begin{bmatrix} 1&1&1&1&0&0 \\ 1&0&0&0&1&1 \\ 0&1&0&0&1&1 \\ 0&0&1&0&1&1 \\ 0&0&0&1&1&1 \end{bmatrix}$$

For this design the matrix **W** has two different eigenvalues ($\alpha_1 = 2.5$, $\nu_1 = 4$; $\alpha_2 = 0$, $\nu_2 = n - a - b + 1 = 6$) and in consequence the UMPIT exists. In Table 3 the power function $p(\theta)$ of the UMPI test is presented.

As we can see the behaviour of it is quite different than in former cases. The function goes to 1 very fast if θ tends to infinity.

Table 3
The power function of the UMPI test for example 3

θ	$p(\theta)$
.001	.0503
.010	.0526
.100	.0782
.500	.2114
1.000	.3693
5.000	.8448
10.000	.9436
50.000	.9968
100.000	.9992
500.000	.9999
∞	1.0000

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Received 15 July 1992; revised 28 November 1992